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Confirmation as partial entailment: A representation theorem in inductive logic

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ABSTRACT

The most prominent research program in inductive logic – here just labeled *The Program*, for simplicity – relies on probability theory as its main building block and aims at a proper generalization of deductive-logical relations by a theory of partial entailment. We prove a representation theorem by which a class of ordinally equivalent measures of *inductive support* or *confirmation* is singled out as providing a uniquely coherent way to work out these two major sources of inspiration of The Program.

The current state of inductive logic may appear puzzling. Some highly sophisticated observers in philosophy, for instance, have come to see the very term as "slightly antiquated" (see [32, p. 291]). Yet the central issue of inductive logic – i.e., the evaluation of how given premises or data affect the credibility of conclusions or hypotheses of interest – never ceased to play a significant role in a wide range of research domains. Up to recent times, striking examples arise from fields such as cognitive psychology, computer science and the law (by way of illustration, see [19], [2], and [1], respectively). Thus, the *problem* of inductive logic seems not to have lost its relevance, which provides motivation to stick to the label after all, whatever its fate in certain philosophical quarters.

Survey presentations usually agree on one account, i.e., that much contemporary work in inductive logic has consistently relied on two pillars. First, *probability* (in its modern mathematical meaning) is viewed as the main "building block" for inductive-logical theorizing. And second, inductive logic is meant to provide an analogue of classical deductive logic in some suitable sense (see [12] and [21]). For the sake of convenience, we will simply use *The Program* to denote the combination of these two guidelines in inductive logic research.

In this contribution, we do not mean to defend The Program as such. We will rather enrich it through a novel formal result, i.e., a representation theorem by which a class of ordinally equivalent measures of *inductive support* or *confirmation* is singled out as capturing a small number of axioms. These axioms, we will argue, provide an unusually neat instantiation of the spirit of The Program itself.

1. Induction and probability

Broadly speaking, the case for the probabilistic side of The Program is pretty straightforward and runs more or less as follows. It is a platitude that induction arises in the presence of uncertainty, and probability is widely recognized as the formal

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representation of uncertainty that is best understood and motivated. (For an updated survey of the main alternative options available, though, see [28].) In order to exploit this point in more detail, we will now need a few technical preliminaries.

Let L be a propositional language. To ensure mathematical definiteness we will focus on the set L_c of the contingent formulas in L (i.e., those expressing neither logical truths nor logical falsehoods) and on the set $\mathbf{P}$ of all *regular* probability functions that can be defined over L, so that for any $\alpha \in L_c$ and $P \in \mathbf{P}$, $0 < P(\alpha) < 1$. Each element $P \in \mathbf{P}$ can now be seen as representing a possible (non-dogmatic, see [26, p. 70]) state of belief concerning a domain described in L. We will posit a function $C : \{L_c \times L_c \times \mathbf{P}\} \rightarrow \Re$ as representing the fundamental inductive-logical relation of support or confirmation and adopt the notation $C_P(h, e)$, with $e, h \in L_c$ denoting the premise (or the conjunction of a collection of premises) and the conclusion of an inductive argument, respectively.[1] Our first axiom will then be as follows:

A0 *(Formality).* There exists a function g such that, for any $e, h \in L_c$ and $P \in \mathbf{P}$, $C_P(h, e) = g[P(h \wedge e), P(h), P(e)]$.

Note that the probability distribution over the algebra generated by e and h is entirely determined by $P(h \wedge e)$, $P(h)$ and $P(e)$. Hence A0 simply states that $C_P(h, e)$ depends on that distribution, and nothing else. This is a widespread (although often tacit) assumption in discussions of induction in a probabilistic framework. From Keynes and Carnap onwards, theorists pursuing The Program are bound to subscribe to A0 more or less as a matter of course. When prompted by technical reasons, moreover, inductive logicians working under the heading of Bayesian confirmation theory (or other related labels) have expressed explicit endorsement of it.[2]

Now consider the following:

A1 *(Final probability incrementality).* For any $e_1, e_2, h \in L_c$ and any $P \in \mathbf{P}$, $C_P(h, e_1) \gtreqless C_P(h, e_2)$ iff $P(h|e_1) \gtreqless P(h|e_2)$.

A1 states that, for any conclusion h, inductive support is an increasing function of the posterior probability conditional on the premise (or conjunction of premises) e at issue. To the best of our knowledge, this also counts as virtually unchallenged an assumption in probabilistic analyses of inductive inference.[3] Notably, it already conveys a minimal form of alignment between inductive and deductive logic. For, if violations of A1 are allowed, then one might have cases in which $e_1 \vDash h$ while $e_2 \nvDash h$, so that $P(h|e_1) = 1 > P(h|e_2)$, and yet $C_P(h, e_1) < C_P(h, e_2)$ (see [52, p. 109] for an example). We will now have to tackle this point in a more thorough and general fashion.

2. Partial entailment – taken seriously

What we have called The Program of inductive logic research has been pursued in a number of variants, mostly depending, as James Hawthorne has observed, on "precisely how the deductive model is emulated" [21]. Our current proposal amounts to downright revival of an old and illustrious idea. According to this view, inductive logic should parallel the deductive model by providing a generalized, quantitative theory of *partial entailment*.[4] The following revealing passage, again from [21], attests to the enduring influence of this notion, albeit in a pessimistic vein:

> A collection of premise sentences *logically entails* a conclusion sentence just when the negation of the conclusion is *logically inconsistent* with those premises. An inductive logic must, it seems, deviate from this paradigm [...]. Although the notion of *inductive support* is analogous to the deductive notion of *logical entailment*, and is arguably an extension of it, there seems to be no inductive logic extension of the notion of *logical inconsistency* – at least none that is interdefinable with *inductive support* in the way that *logical inconsistency* is interdefinable with *logical entailment*. (All italics in the original.)

A central goal of our discussion here is to show that this resignation is hasty. It is perfectly possible, we urge, to have a sound inductive-logical extension of the notion of logical inconsistency that is indeed interdefinable with inductive support in essentially the same way that logical inconsistency is interdefinable with logical entailment. So much so, we submit, that one can safely and fruitfully embed into axioms those very properties that inductive logic would inevitably lack according to Hawthorne.

First, we will assume the inductive-logical measure $C_P(h, e)$ to exhibit a commutative behavior whenever e and h are inductively at odds (i.e., negatively correlated), thus paralleling the symmetric nature of logical inconsistency, as follows:

A2 *(Partial inconsistency).* For any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h \wedge e) \leqslant P(h)P(e)$, then $C_P(h, e) = C_P(e, h)$.

[1] To allow for relevant background knowledge and assumptions, a further term B should be included, thus having $C_P(h, e|B)$. Such a term will be omitted from our notation for simple reasons of convenience, as it is inconsequential for our discussion.

[2] See [15, p. 322], [16, pp. 127–128], and [36, p. 21]. The label *formality* is taken from [53,54].

[3] Relevant occurrences of A1 or closely related principles include the following: [5, pp. 77–80], [8, p. 670], [10, p. 58], [13, p. 506], [17, p. 295], [20, p. 122], [22], [25, p. 53], [50, pp. 219–221], and [51, p. 60].

[4] The idea of partial entailment can be shown to reach back to [31] and [3]. For the label, however, [44] is a key reference.

An unrestricted form of commutativity has appeared as a basic and sound requirement in probabilistic analyses of degrees of "coherence" (and lack thereof).[5] In A2, however, commutativity is not meant to extend to the quantification of *positive* confirmation or support, because logical entailment (unlike refutation) is not symmetric, nor is it coextensive with logical equivalence (or mere logical consistency, for that matter) in the way that refutation is coextensive with inconsistency (Eells and Fitelson [9] and Crupi et al. [7] discuss this point further).

As for the interdefinability of logical entailment of h from e and inconsistency of e with $\neg h$, it naturally generalizes to an inverse (ordinal) correlation between positive inductive support and partial inconsistency with regard to complementary conclusions, as follows:

A3 *(Complementarity).* For any $e, h_1, h_2 \in L_c$ and any $P \in \mathbf{P}$, $C_P(h_1, e) \gtreqless C_P(h_2, e)$ iff $C_P(\neg h_1, e) \lesseqgtr C_P(\neg h_2, e)$.

A3 can be seen as a fairly faithful formal rendition of Keynes's remark that "an argument is always as near to proving or disproving a proposition, as it is to disproving or proving its contradictory" [31, p. 80]. Indeed, A3 has been put forward by several theorists, including other leading figures such as Carnap, sometimes in the stronger form of some specific functional dependency between $C_P(h, e)$ and $C_P(\neg h, e)$, like $C_P(h, e) = -C_P(\neg h, e)$ (see [3, §67]; also see [7, pp. 238–239], [9, p. 134], and [30, p. 309]). Here, however, we prefer to stick to the ordinal/comparative level as a firmer basis for an axiomatic approach.

To sum up, A2 implies that, when e and h are at odds, the central inductive-logical function C_P in fact amounts to a measure of their partial inconsistency. A3, on the other hand, implies that the positive inductive support from e to h is in fact nothing other than a strictly decreasing function of the degree of partial inconsistency between e and $\neg h$. Hawthorne's [21] "impossibility" claim above would suggest that no sensible inductive-logical theory could satisfy such requirements, no matter how appealing they may seem from within The Program. By stating them as axioms, we are intentionally turning this line of argument upside down. Let us see what follows.

3. A representation theorem

The following can be proved (see Appendix A):

Theorem. A0–A3 *if and only if there exists a strictly increasing function f such that $C_P(h, e) = f[z(h, e)]$, where*

$$z(h,e) = \begin{cases} \frac{P(h|e)-P(h)}{1-P(h)} & \text{if } P(h|e) \geqslant P(h), \\ \frac{P(h|e)-P(h)}{P(h)} & \text{if } P(h|e) < P(h). \end{cases}$$

$z(h, e)$ itself is a particularly appealing exemplar of the class of (ordinal) equivalence singled out by the theorem, especially for its neatly symmetrical and bounded range $[-1, +1]$. Also, despite its twofold algebraic form, it conveys a unifying core intuition.[6]

To appreciate this conceptual unity, note that in case of positive inductive support or confirmation $z(h, e)$ expresses the relative reduction of the initial distance from certainty of h being true as yielded by e, i.e., it measures how far upward the posterior $P(h|e)$ has gone in covering the distance between the prior $P(h)$ and 1. Similarly, in the case of negative inductive support or disconfirmation, $z(h, e)$ reflects the relative reduction of the initial distance from certainty of h being false as yielded by e, i.e., it measures how far downward the posterior $P(h|e)$ has gone in covering the distance between the prior $P(h)$ and 0. Accordingly, $z(h, e)$ measures the extent to which the initial probability distance from certainty concerning the truth (falsehood) of h is reduced by the confirming (disconfirming) statement e. Or, put otherwise, how much of such distance is "covered" by the upward (downward) jump from $P(h)$ to $P(h|e)$. Thus, $z(h, e)$ is a measure of the relative reduction of the distance from certainty that a conclusion/hypothesis of interest is true or false – or, with a slight abuse of language, a *relative distance* measure. (See [5], for a more extensive discussion. The "relative distance" label was first adopted by Huber [27], while reporting on [7].)

Interestingly, $z(h, e)$ is not an entirely new idea. Sparse occurrences can be found in rather diverse settings, such as the formal analysis of *certainty factors*, a central notion to represent uncertain reasoning in early expert systems (see [48] and [23]). Apart from [7], the only further appearance in the philosophical literature seems to be in [42, pp. 86–87] (where a

[5] See Shogenji's [47] seminal work. For an updated and informed discussion of subsequent developments, see [46]. For a neat investigation on a *non*-probabilistic extension of logical inconsistency, see [4].

[6] An alternative, more compact rendition is the following:

$$z(h,e) = \frac{\min[P(h|e), P(h)]}{P(h)} - \frac{\min[P(\neg h|e), P(\neg h)]}{P(\neg h)}.$$

In this form, $z(h, e)$ is structurally similar to Mura's [37,38] measure of "partial entailment". Mura's measure and $z(h, e)$, however, are demonstrably non-equivalent in ordinal terms.

different measure is eventually endorsed, though). It should also be mentioned, however, that the positive branch of $z(h,e)$ is ordinally equivalent to a confirmation measure proposed by Gaifman [14, p. 120], is identical to a measure of *inductive strength* mentioned by Rips [43, p. 129, fn. 1], and has been given further attention recently by Schlosshauer and Wheeler [45], and Wheeler and Scheines [56,57].

To the extent that relative distance measures – i.e., measures that are ordinally equivalent to $z(h,e)$ – represent a coherent probabilistic generalization of deductive-logical relationships, one might well expect to find other suitable connections. *Contraposition* offers an effective illustration. The deductive-logical principle of contraposition says that e implies h if and only if $\neg h$ implies $\neg e$. A straightforward inductive-logical extension would state that $C_P(h,e) = C_P(\neg e, \neg h)$, provided that positive inductive support or confirmation is at issue. The latter caveat is important, for contraposition does not apply to refutation, i.e., it is not the case that e (deductively) refutes h if and only if $\neg h$ refutes $\neg e$. (Example: we have a disproving argument from "the card randomly drawn is a 7" to "the card drawn is a picture"; not so from "the card drawn is not a picture" to "the card drawn is not a 7".) Indeed, relative distance measures do imply the relevant principle, i.e., the following (see [7]):

A4 *(Inductive-logical contraposition).* For any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h \wedge e) \geqslant P(h)P(e)$, then $C_P(h,e) = C_P(\neg e, \neg h)$.

One can even prove that, if A3 is assumed, then A2 and A4 are logically equivalent. Interestingly, this also means that they are entirely interchangeable in the statement of our representation theorem.

4. Discussion

Let us go back to Hawthorne's [21] quote above. In his treatment, a pessimistic conclusion arises from one specific version of The Program, i.e., the popular idea of probability *per se* as the core inductive-logical notion.[7] For, if one posits $C_P(h,e) = f[P(h|e)]$ (with f a strictly increasing function), then A0–A1 are satisfied, while Hawthorne's "deviations" from the deductive-logical paradigm concurrently emerge. The strongest conclusion that inductive logic "must" exhibit such deviations fails, however. One feature that allows relative distance measures to do the trick (with further consequences along the very same line, see A4) is that they represent inductive support in terms of *impact* via probabilistic *relevance*, not overall *credibility* via probability as such. On this account, our analysis has tacitly followed John Irving Good's remark that "if you had $P(h|e)$ close to unity, but less than $P(h)$, you *ought not* to say that h was confirmed by e" [16, p. 134]. Interestingly, this clear-cut distinction between posterior probability and inductive support or confirmation has proved recurrently necessary for theoretical clarity in philosophy as well as in artificial intelligence and the psychology of reasoning (see [6,24,40,41,55]).

Measures of confirmation *qua* probabilistic relevance are well known to be many and diverse, and have been said to "capture distinct, complementary notions of evidential support" [20, p. 123]. Whatever the amount of pluralism that one is willing to allow for in this respect, our result shows that a small set of properties singles out relative distance measures as uniquely capturing the notion of partial entailment. Some very influential alternatives (like the probability difference and likelihood ratio measures) do themselves satisfy A0–A1, but face the same limitations emerging from Hawthorne's remarks. On the other hand, there does exist a confirmation measure devised and analyzed by Carnap [3, §67] – i.e., $P(h \wedge e) - P(h)P(e)$ – which overcomes the latter limitations in that it satisfies A2 and A3. Yet it demonstrably violates the pivotal principle A1 (see [11] for a proof).

To conclude, what we have called The Program of inductive logic research has elicited varying degrees of confidence, effort and determination. While it has found a good deal of criticism, it seems to have remained prominent nonetheless, and stands as a deserving theoretical endeavor at least in our view.[8] Be that as it may, we point out that relative distance measures represent a uniquely coherent way to combine the reliance on a probabilistic analysis along with the aim at a proper generalization of deductive-logical relations by a theory of partial entailment. These being the main sources of inspiration of The Program itself, such a result could meet the interest of its advocates and critics alike.

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[7] In fact, the passage continues as follows: "*e logically entails h* just when $e \wedge \neg h$ is *logically inconsistent*. However, it turns out that when the unconditional probability of $e \wedge \neg h$ is nearly 0 (i.e., when $e \wedge \neg h$ is 'nearly inconsistent'), the degree to which *e inductively supports h*, $P(h|e)$, may range anywhere from nearly 0 to very near 1" (notation adjusted, all italics in the original).

[8] For a small sample of recent work in the spirit of The Program, see [13,18,34,49]. For a taste of diverse but comparably forceful critical voices, see [29,35,39]. Of course, there are cases which tend to defy the distinction between representatives and critics: Isaac Levi's work (e.g., [33]) is a major example.

Appendix A

Theorem. A0–A3 *if and only if there exists a strictly increasing function* f *such that* $C_P(h,e) = f[z(h,e)]$, *where*

$$z(h,e) = \begin{cases} \frac{P(h|e)-P(h)}{1-P(h)} & \text{if } P(h|e) \geqslant P(h), \\ \frac{P(h|e)-P(h)}{P(e)} & \text{if } P(h|e) < P(h). \end{cases}$$

Proof. *Right-to-left implication*

A0. If there exists a strictly increasing function f such that $C_P(h,e) = f[z(h,e)]$, then A0 is trivially satisfied.

A1. Let $e_1, e_2, h \in L_c$ be given. Three classes of cases can obtain. (i) Let $P \in \mathbf{P}$ be such that $P(h|e_1) \gtreqless P(h) \gtreqless P(h|e_2)$. It is easy to verify that, for any $e, h \in L_c$, $P(h|e) \gtreqless P(h)$ iff $z(h,e) \gtreqless 0$. So we have that, for any $e_1, e_2, h \in L_c$, $P(h|e_1) \gtreqless P(h)$ iff $z(h,e_1) \gtreqless 0$ and $P(h|e_2) \lesseqgtr P(h)$ iff $z(h,e_2) \lesseqgtr 0$. It follows that, for any $e_1, e_2, h \in L_c$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $z(h,e_1) \gtreqless z(h,e_2)$. (ii) Let $P \in \mathbf{P}$ be such that $P(h|e_1), P(h|e_2) \geqslant P(h)$. Then we have that, for any $e_1, e_2, h \in L_c$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $P(\neg h|e_1) \lesseqgtr P(\neg h|e_2)$ iff $P(\neg h|e_1)/P(\neg h) \lesseqgtr P(\neg h|e_2)/P(\neg h)$ iff $1 - P(\neg h|e_1)/P(\neg h) \gtreqless 1 - P(\neg h|e_2)/P(\neg h)$ iff $z(h,e_1) \gtreqless z(h,e_2)$. (iii) Finally, let $P \in \mathbf{P}$ be such that $P(h|e_1), P(h|e_2) \leqslant P(h)$. Then we have that, for any $e_1, e_2, h \in L_c$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $P(h|e_1)/P(h) \gtreqless P(h|e_2)/P(h)$ iff $P(h|e_1)/P(h) - 1 \gtreqless P(h|e_2)/P(h) - 1$ iff $z(h,e_1) \gtreqless z(h,e_2)$. As (i)–(iii) are exhaustive, for any $e_1, e_2, h \in L_c$ and any $P \in \mathbf{P}$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $z(h,e_1) \gtreqless z(h,e_2)$. By ordinal equivalence, if there exists a strictly increasing function f such that $C_P(h,e) = f[z(h,e)]$, then A1 follows.

A2. Let $e, h \in L_c$ and $P \in \mathbf{P}$ be given so that $P(h \wedge e) \leqslant P(h)P(e)$. This is equivalent both to $P(h|e) \leqslant P(h)$ and to $P(e|h) \leqslant P(e)$. Then we have that, for any $e, h \in L_c$, $P(h|e)/P(h) = P(e|h)/P(e)$ iff $P(h|e)/P(h) - 1 = P(e|h)/P(e) - 1$ iff $z(h,e) = z(e,h)$. So for any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h \wedge e) \leqslant P(h)P(e)$, then $z(h,e) = z(e,h)$. By ordinal equivalence, if there exists a strictly increasing function f such that $C_P(h,e) = f[z(h,e)]$, then A2 follows.

A3. Let $e, h_1, h_2 \in L_c$ and $P \in \mathbf{P}$ be given. Three classes of cases can obtain. (i) Let $P \in \mathbf{P}$ be such that $P(h_1|e) \gtreqless P(h_1)$ and $P(h_2|e) \lesseqgtr P(h_2)$. It is easy to verify that, for any $e, h \in L_c$, $P(h|e) \gtreqless P(h)$ iff $z(h,e) \gtreqless 0$ iff $P(\neg h|e) \lesseqgtr P(\neg h)$ iff $z(\neg h,e) \lesseqgtr 0$. So we have that, for any $e, h_1, h_2 \in L_c$, $P(h_1|e) \gtreqless P(h_1)$ iff $z(h_1,e) \gtreqless 0$ iff $P(\neg h_1|e) \lesseqgtr P(\neg h_1)$ iff $z(\neg h_1,e) \lesseqgtr 0$ and $P(h_2|e) \gtreqless P(h_2)$ iff $z(h_2,e) \gtreqless 0$ iff $P(\neg h_2|e) \lesseqgtr P(\neg h_2)$ iff $z(\neg h_2,e) \lesseqgtr 0$. It follows that, for any $e, h_1, h_2 \in L_c$, $z(h_1,e) \gtreqless z(h_2,e)$ iff $z(\neg h_1,e) \lesseqgtr z(\neg h_2,e)$. (ii) Let $P \in \mathbf{P}$ be such that $P(h_1|e) \geqslant P(h_1)$ and $P(h_2|e) \geqslant P(h_2)$. Then we have that, for any $e, h_1, h_2 \in L_c$, $z(h_1,e) \gtreqless z(h_2,e)$ iff $1 - P(\neg h_1|e)/P(\neg h_1) \gtreqless 1 - P(\neg h_2|e)/P(\neg h_2)$ iff $P(\neg h_1|e)/P(\neg h_1) \lesseqgtr P(\neg h_2|e)/P(\neg h_2)$ iff $P(\neg h_1|e)/P(\neg h_1) - 1 \lesseqgtr P(\neg h_2|e)/P(\neg h_2) - 1$ iff $z(\neg h_1,e) \lesseqgtr z(\neg h_2,e)$. (iii) Finally, let $P \in \mathbf{P}$ be such that $P(h_1|e) \leqslant P(h_1)$ and $P(h_2|e) \leqslant P(h_2)$. Then we have that, for any $e, h_1, h_2 \in L_c$, $z(h_1,e) \gtreqless z(h_2,e)$ iff $P(h_1|e)/P(h_1) - 1 \gtreqless P(h_2|e)/P(h_2) - 1$ iff $P(h_1|e)/P(h_1) \gtreqless P(h_2|e)/P(h_2)$ iff $1 - P(h_1|e)/P(h_1) \lesseqgtr 1 - P(h_2|e)/P(h_2)$ iff $z(\neg h_1,e) \lesseqgtr z(\neg h_2,e)$. As (i)–(iii) are exhaustive, for any $e, h_1, h_2 \in L_c$ and any $P \in \mathbf{P}$, $z(h_1,e) \gtreqless z(h_2,e)$ iff $z(\neg h_1,e) \lesseqgtr z(\neg h_2,e)$. By ordinal equivalence, if there exists a strictly increasing function f such that $C_P(h,e) = f[z(h,e)]$, then A3 follows.

Left-to-right implication

The case of disconfirmation $(P(h|e) \leqslant P(h))$

Note that $P(h \wedge e) = [P(h|e)/P(h)]P(h)P(e)$. As a consequence, by A0, there exist a function j such that, for any $e, h \in L_c$ and any $P \in \mathbf{P}$, $C_P(h,e) = j[P(h|e)/P(h), P(h), P(e)]$. With no loss of generality, we will convey probabilistic coherence, regularity and disconfirmation by constraining the domain of j to include triplets of values (x, y, w) such that the following conditions are jointly satisfied:

- $0 < y, w < 1$;
- $x \geqslant 0$, by which $x = P(h|e)/P(h) \geqslant 0$, so that $P(h|e) \geqslant 0$, and thus $P(h \wedge e) \geqslant 0$;
- $x \leqslant 1$ (conveying disconfirmation, i.e., $P(h|e) \leqslant P(h)$), by which $xy = P(h|e) < 1$, so that $P(h \wedge e) < P(e)$, and thus $P(\neg h \wedge e) > 0$, and $xw = P(e|h) < 1$, so that $P(h \wedge e) < P(h)$, and thus $P(h \wedge \neg e) > 0$;
- $x \geqslant (y + w - 1)/yw$, by which $xyw = P(h \wedge e) \geqslant P(h) + P(e) - 1 = y + w - 1$, and thus $P(h \wedge e) + P(\neg h \wedge e) + P(h \wedge \neg e) \leqslant 1$.

We thus posit $j: \{(x, y, w) \in [0,1] \times (0,1)^2 \mid x \geqslant (y + w - 1)/yw\} \rightarrow \Re$ and denote the domain of j as D_j.

Lemma 1. *For any* x, y, w_1, w_2 *such that* $x \in [0,1]$, $y, w_1, w_2 \in (0,1)$, *and* $x \geqslant (y + w_1 - 1)/yw_1$, $(y + w_2 - 1)/yw_2$, *there exist* $e_1, e_2, h \in L_c$ *and* $P' \in \mathbf{P}$ *such that* $P'(h|e_1)/P'(h) = P'(h|e_2)/P'(h) = x$, $P'(h) = y$, $P'(e_1) = w_1$, *and* $P'(e_2) = w_2$.

Proof. The equalities in Lemma 1 arise from the following scheme of probability assignments:

$$P'(h\wedge e_1\wedge e_2)=(xw_1)(xw_2)y;\qquad P'(\neg h\wedge e_1\wedge e_2)=\tfrac{(1-xy)^2w_1w_2}{(1-y)};$$
$$P'(h\wedge e_1\wedge\neg e_2)=(xw_1)(1-xw_2)y;\qquad P'(\neg h\wedge e_1\wedge\neg e_2)=(1-xy)w_1\big[1-\tfrac{(1-xy)w_2}{(1-y)}\big];$$
$$P'(h\wedge\neg e_1\wedge e_2)=(1-xw_1)(xw_2)y;\qquad P'(\neg h\wedge\neg e_1\wedge e_2)=\big[1-\tfrac{(1-xy)w_1}{(1-y)}\big](1-xy)w_2;$$
$$P'(h\wedge\neg e_1\wedge\neg e_2)=(1-xw_1)(1-xw_2)y;\qquad P'(\neg h\wedge\neg e_1\wedge\neg e_2)=\big[1-\tfrac{(1-xy)w_1}{(1-y)}\big]\big[1-\tfrac{(1-xy)w_2}{(1-y)}\big](1-y).$$

Suppose there exist $(x,y,w_1),(x,y,w_2)\in D_j$ such that $j(x,y,w_1)\neq j(x,y,w_2)$. Then, by Lemma 1 and the definition of D_j, there exist $e_1,e_2,h\in L_c$ and $P'\in\mathbf{P}$ such that $P'(h|e_1)/P'(h)=P'(h|e_2)/P'(h)=x$, $P'(h)=y$, $P'(e_1)=w_1$, and $P'(e_2)=w_2$. Clearly, if the latter equalities hold, then $P'(h|e_1)=P'(h|e_2)$. Thus, there exist $e_1,e_2,h\in L_c$ and $P'\in\mathbf{P}$ such that $C_{P'}(h,e_1)=j(x,y,w_1)\neq j(x,y,w_2)=C_{P'}(h,e_2)$ even if $P'(h|e_1)=P'(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $(x,y,w_1),(x,y,w_2)\in D_j$, $j(x,y,w_1)=j(x,y,w_2)$. So, for A1 to hold, there must exist k such that, for any $e,h\in L_c$ and any $P\in\mathbf{P}$, if $P(h|e)\leqslant P(h)$, then $C_P(h,e)=k[P(h|e)/P(h),P(h)]$ and $k(x,y)=j(x,y,w)$. We thus posit $k:\{(x,y)\in[0,1]\times(0,1)\}\to\Re$ and denote the domain of k as D_k.

Lemma 2. *For any x,y_1,y_2 such that $x\in[0,1]$ and $y_1,y_2\in(0,1)$, there exist $e,h\in L_c$ and $P''\in\mathbf{P}$ such that $P''(h|e)/P''(h)=P''(e|h)/P''(e)=x$, $P''(h)=y_1$, and $P''(e)=y_2$.*

Proof. The equalities in Lemma 2 arise from the following scheme of probability assignments:

$$P''(h\wedge e)=xy_1y_2;\qquad P''(\neg h\wedge e)=(1-xy_1)y_2;$$
$$P''(h\wedge\neg e)=(1-xy_2)y_1;\qquad P''(\neg h\wedge\neg e)=(1-y_1)-(1-xy_1)y_2.$$

Suppose there exist $(x,y_1),(x,y_2)\in D_k$ such that $k(x,y_1)\neq k(x,y_2)$. Then, by Lemma 2 and the definition of D_k, there exist $e,h\in L_c$ and $P''\in\mathbf{P}$ such that $P''(h|e)/P''(h)=P''(e|h)/P''(e)=x$, $P''(h)=y_1$, and $P''(e)=y_2$. By the probability calculus, if the latter equalities hold, then $P''(h\wedge e)\leqslant P''(h)P''(e)$. Thus, there exist $e,h\in L_c$ and $P'\in\mathbf{P}$ such that $C_{P''}(h,e)=k(x,y_1)\neq k(x,y_2)=C_{P''}(e,h)$ even if $P''(h\wedge e)\leqslant P''(h)P''(e)$, contradicting A2. Conversely, A2 implies that, for any $(x,y_1),(x,y_2)\in D_k$, $k(x,y_1)=k(x,y_2)$. So, for A2 to hold, there must exist k such that, for any $e,h\in L_c$ and any $P\in\mathbf{P}$, if $P(h|e)\leqslant P(h)$, then $C_P(h,e)=m[P(h|e)/P(h)]$ and $m(x)=k(x,y)$. We thus posit $m:[0,1]\to\Re$ and denote the domain of m as D_m.

Lemma 3. *For any $x_1,x_2\in[0,1]$, there exist $e_1,e_2,h\in L_c$ and $P'''\in\mathbf{P}$ such that $P'''(h|e_1)/P'''(h)=x_1$ and $P'''(h|e_2)/P'''(h)=x_2$.*

Proof. Let $y,w_1,w_2\in(0,1)$ be given so that $w_1\leqslant(1-y)/(1-x_1y)$ (as the latter quantity must be positive, w_1 exists), and $w_2\leqslant(1-y)/(1-x_2y)$ (as the latter quantity must be positive, w_2 exists). The equalities in Lemma 3 arise from the following scheme of probability assignments:

$$P'''(h\wedge e_1\wedge e_2)=(x_1w_1)(x_2w_2)y;\qquad P'''(\neg h\wedge e_1\wedge e_2)=\tfrac{(1-x_1y)(1-x_2y)w_1w_2}{(1-y)};$$
$$P'''(h\wedge e_1\wedge\neg e_2)=(x_1w_1)(1-x_2w_2)y;\qquad P'''(\neg h\wedge e_1\wedge\neg e_2)=(1-x_1y)w_1\big[1-\tfrac{(1-x_2y)w_2}{(1-y)}\big];$$
$$P'''(h\wedge\neg e_1\wedge e_2)=(1-x_1w_1)(x_2w_2)y;\qquad P'''(\neg h\wedge\neg e_1\wedge e_2)=\big[1-\tfrac{(1-x_1y)w_1}{(1-y)}\big](1-x_2y)w_2;$$
$$P'''(h\wedge\neg e_1\wedge\neg e_2)=(1-x_1w_1)(1-x_2w_2)y;\qquad P'''(\neg h\wedge\neg e_1\wedge\neg e_2)=\big[1-\tfrac{(1-x_1y)w_1}{(1-y)}\big]\big[1-\tfrac{(1-x_2y)w_2}{(1-y)}\big](1-y).$$

Suppose there exist $x_1,x_2\in D_m$ such that $x_1>x_2$ and $m(x_1)\leqslant m(x_2)$. Then, by Lemma 3 and the definition of D_m, there exist $e_1,e_2,h\in L_c$ and $P'''\in\mathbf{P}$ such that $P'''(h|e_1)/P'''(h)=x_1$ and $P'''(h|e_2)/P'''(h)=x_2$. Clearly, if the latter equalities hold, then $P'''(h|e_1)>P'''(h|e_2)$. Thus, there exist $e_1,e_2,h\in L_c$ and $P'''\in\mathbf{P}$ such that $C_{P'''}(h,e_1)=m(x_1)\leqslant m(x_2)=C_{P'''}(h,e_2)$ even if $P'''(h|e_1)>P'''(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $x_1,x_2\in D_m$, if $x_1>x_2$ then $m(x_1)>m(x_2)$. By a similar argument, A1 also implies that, for any $x_1,x_2\in D_m$, if $x_1=x_2$ then $m(x_1)=m(x_2)$. So, for A1 to hold, it must be that, for any $e,h\in L_c$ and any $P\in\mathbf{P}$, if $P(h|e)\leqslant P(h)$, then $C_P(h,e)=m[P(h|e)/P(h)]$ and m is a strictly increasing function.

The case of confirmation $(P(h|e)>P(h))$

Notice that $P(h\wedge e)=[1-\frac{P(\neg h|e)}{P(\neg h)}P(\neg h)]P(e)$ and $P(h)=1-P(\neg h)$. As a consequence, by A0, there exist a function r such that, for any $e,h\in L_c$ and any $P\in\mathbf{P}$, $C_P(h,e)=r[P(\neg h|e)/P(\neg h),P(\neg h),P(e)]$. With no loss of generality, we will convey probabilistic coherence, regularity and confirmation by constraining the domain of r to include triplets of values (x,y,w) such that the following conditions are jointly satisfied:

- $0<y,\ w<1$;
- $x\geqslant 0$, by which $x=P(\neg h|e)/P(\neg h)\geqslant 0$, so that $P(\neg h|e)\geqslant 0$, and thus $P(\neg h\wedge e)\geqslant 0$;

- $x < 1$ (conveying confirmation, i.e., $P(h|e) > P(h)$), by which $xy = P(\neg h|e) < 1$, so that $P(\neg h \wedge e) < P(e)$, and thus $P(h \wedge e) > 0$, and $xw = P(e|\neg h) < 1$, so that $P(\neg h \wedge e) < P(\neg h)$, and thus $P(\neg h \wedge \neg e) > 0$;
- $x \geqslant (y + w - 1)/yw$, by which $xyw = P(\neg h \wedge e) \geqslant P(\neg h) + P(e) - 1 = y + w - 1$, and thus $P(\neg h \wedge e) + P(h \wedge e) + P(\neg h \wedge \neg e) \leqslant 1$.

We thus posit $r : \{(x, y, w) \in [0, 1) \times (0, 1)^2 \mid x \geqslant (y + w - 1)/yw\} \to \Re$ and denote the domain of r as D_r.

Lemma 4. *For any x, y, w_1, w_2 such that $x \in [0, 1)$, $y, w_1, w_2 \in (0, 1)$, and $x \geqslant (y + w_1 - 1)/yw_1$, $(y + w_2 - 1)/yw_2$, there exist $e_1, e_2, h \in L_c$ and $P' \in \mathbf{P}$ such that $P'(\neg h|e_1)/P'(\neg h) = P'(\neg h|e_2)/P'(\neg h) = x$, $P'(\neg h) = y$, $P'(e_1) = w_1$, and $P'(e_2) = w_2$.*

Proof. The equalities in Lemma 4 arise from the following scheme of probability assignments:

$$P'(h \wedge e_1 \wedge e_2) = \frac{(1-xy)^2 w_1 w_2}{(1-y)}; \qquad P'(\neg h \wedge e_1 \wedge e_2) = (xw_1)(xw_2)y;$$
$$P'(h \wedge e_1 \wedge \neg e_2) = (1 - xy)w_1\Big[1 - \frac{(1-xy)w_2}{(1-y)}\Big]; \qquad P'(\neg h \wedge e_1 \wedge \neg e_2) = (xw_1)(1 - xw_2)y;$$
$$P'(h \wedge \neg e_1 \wedge e_2) = \Big[1 - \frac{(1-xy)w_1}{(1-y)}\Big](1 - xy)w_2; \qquad P'(\neg h \wedge \neg e_1 \wedge e_2) = (1 - xw_1)(xw_2)y;$$
$$P'(h \wedge \neg e_1 \wedge \neg e_2) = \Big[1 - \frac{(1-xy)w_1}{(1-y)}\Big]\Big[1 - \frac{(1-xy)w_2}{(1-y)}\Big](1 - y); \qquad P'(\neg h \wedge \neg e_1 \wedge \neg e_2) = (1 - xw_1)(1 - xw_2)y.$$

Suppose there exist $(x, y, w_1), (x, y, w_2) \in D_r$ such that $r(x, y, w_1) \neq r(x, y, w_2)$. Then, by Lemma 4 and the definition of D_r, there exist $e_1, e_2, h \in L_c$ and $P' \in \mathbf{P}$ such that $P'(\neg h|e_1)/P'(\neg h) = P'(\neg h|e_2)/P'(\neg h) = x$, $P'(\neg h) = y$, $P'(e_1) = w_1$, and $P'(e_2) = w_2$. By the probability calculus, if the latter equalities hold, then $P'(h|e_1) = P'(h|e_2)$. Thus, there exist $e_1, e_2, h \in L_c$ and $P' \in \mathbf{P}$ such that $C_{P'}(h, e_1) = r(x, y, w_1) \neq r(x, y, w_2) = C_{P'}(h, e_2)$ even if $P'(h|e_1) = P'(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $(x, y, w_1), (x, y, w_2) \in D_r$, $r(x, y, w_1) = r(x, y, w_2)$. So, for A1 to hold, there must exist s such that, for any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h|e) > P(h)$, then $C_P(h, e) = s[P(\neg h|e)/P(\neg h), P(\neg h)]$ and $s(x, y) = r(x, y, w)$. We thus posit $s : \{(x, y) \in [0, 1) \times (0, 1)\} \to \Re$ and denote the domain of s as D_s.

Lemma 5. *For any x, y_1, y_2 such that $x \in [0, 1)$ and $y_1, y_2 \in (0, 1)$, there exist $e, h_1, h_2 \in L_c$ and $P'' \in \mathbf{P}$ such that $P''(\neg h_1|e)/P''(\neg h_1) = P''(\neg h_2|e)/P''(\neg h_2) = x$, $P''(\neg h_1) = y_1$, and $P''(\neg h_2) = y_2$.*

Proof. Let $w \in (0, 1)$ be given so that $w \leqslant (1 - y_1)/(1 - xy_1), (1 - y_2)/(1 - xy_2)$ (as both latter quantities must be positive, w exists). The equalities in Lemma 5 arise from the following scheme of probability assignments:

$$P''(h_1 \wedge h_2 \wedge e) = (1 - xy_1)(1 - xy_2)w; \qquad P''(\neg h_1 \wedge h_2 \wedge e) = (xy_1)(1 - xy_2)w;$$
$$P''(h_1 \wedge h_2 \wedge \neg e) = \Big[1 - \frac{(1-xw)y_1}{(1-w)}\Big]\Big[1 - \frac{(1-xw)y_2}{(1-w)}\Big](1 - w); \qquad P''(\neg h_1 \wedge h_2 \wedge \neg e) = (1 - xw)y_1\Big[1 - \frac{(1-xw)y_2}{(1-w)}\Big];$$
$$P''(h_1 \wedge \neg h_2 \wedge e) = (1 - xy_1)(xy_2)w; \qquad P''(\neg h_1 \wedge \neg h_2 \wedge e) = (xy_1)(xy_2)w;$$
$$P''(h_1 \wedge \neg h_2 \wedge \neg e) = \Big[1 - \frac{(1-xw)y_1}{(1-w)}\Big](1 - xw)y_2; \qquad P''(\neg h_1 \wedge \neg h \wedge_2 \neg e) = \frac{(1-xw)^2 y_1 y_2}{(1-w)}.$$

Suppose there exist $(x, y_1), (x, y_2) \in D_s$ such that $s(x, y_1) \neq s(x, y_2)$. Then, by Lemma 5 and the definition of D_s, there exist $e, h_1, h_2 \in L_c$ and $P'' \in \mathbf{P}$ such that $P''(\neg h_1|e)/P''(\neg h_1) = P''(\neg h_2|e)/P''(\neg h_2) = x$, $P''(\neg h_1) = y_1$, $P''(\neg h_2) = y_2$. If the latter equalities hold, then $C_{P''}(\neg h_1, e) = m[P''(\neg h_1|e)/P''(\neg h_1)] = m[P''(\neg h_2|e)/P''(\neg h_2)] = C_{P''}(\neg h_2, e)$. Thus, there exist $e, h_1, h_2 \in L_c$ and $P'' \in \mathbf{P}$ such that $C_{P''}(h_1, e) = s(x, y_1) \neq s(x, y_2) = C_{P''}(h_1, e)$ even if $C_{P''}(\neg h_1, e) = C_{P''}(\neg h_2, e)$, contradicting A3. So, for A3 to hold, there must exist t such that, for any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h|e) > P(h)$, then $C_P(e, h) = t[P(\neg h|e)/P(\neg h)]$ and $t(x) = s(x, y)$. We thus posit $t : [0, 1) \to \Re$ and denote the domain of t as D_t.

Lemma 6. *For any $x_1, x_2 \in [0, 1)$, there exist $e_1, e_2, h \in L_c$ and $P''' \in \mathbf{P}$ such that $P'''(\neg h|e_1)/P'''(\neg h) = x_1$ and $P'''(\neg h|e_2)/P'''(\neg h) = x_2$.*

Proof. Let $y, w_1, w_2 \in (0, 1)$ be given so that $w_1 \leqslant (1 - y)/(1 - x_1 y)$ (as the latter quantity must be positive, w_1 exists) and $w_2 \leqslant (1 - y)/(1 - x_2 y)$ (as the latter quantity must be positive, w_2 exists). The equalities in Lemma 6 arise from the following scheme of probability assignments:

$$P'''(h \wedge e_1 \wedge e_2) = \frac{(1-x_1 y)(1-x_2 y)w_1 w_2}{(1-y)}; \qquad P'''(\neg h \wedge e_1 \wedge e_2) = (x_1 w_1)(x_2 w_2)y;$$
$$P'''(h \wedge e_1 \wedge \neg e_2) = (1 - x_1 y)w_1\Big[1 - \frac{(1-x_2 y)w_2}{(1-y)}\Big]; \qquad P'''(\neg h \wedge e_1 \wedge \neg e_2) = (x_1 w_1)(1 - x_2 w_2)y;$$
$$P'''(h \wedge \neg e_1 \wedge e_2) = \Big[1 - \frac{(1-x_1 y)w_1}{(1-y)}\Big](1 - x_2 y)w_2; \qquad P'''(\neg h \wedge \neg e_1 \wedge e_2) = (1 - x_1 w_1)(x_2 w_2)y;$$
$$P'''(h \wedge \neg e_1 \wedge \neg e_2) = \Big[1 - \frac{(1-x_1 y)w_1}{(1-y)}\Big]\Big[1 - \frac{(1-x_2 y)w_2}{(1-y)}\Big](1 - y); \qquad P'''(\neg e_1 \wedge \neg e_2 \wedge \neg h) = (1 - x_1 w_1)(1 - x_2 w_2)y.$$

Suppose there exist $x_1, x_2 \in D_t$ such that $x_1 > x_2$ and $t(x_1) \geqslant t(x_2)$. Then, by Lemma 6 and the definition of D_t, there exist $e_1, e_2, h \in L_c$ and $P''' \in \mathbf{P}$ such that $P'''(\neg h|e_1)/P'''(\neg h) = x_1$ and $P'''(\neg h|e_2)/P'''(\neg h) = x_2$. By the probability calculus, if the latter equalities hold, then $P'''(h|e_1) < P'''(h|e_2)$. Thus, there exist $e_1, e_2, h \in L_c$ and $P''' \in \mathbf{P}$ such that $C_{P'''}(h, e_1) = t(x_1) \geqslant t(x_2) = C_{P'''}(h, e_2)$ even if $P'''(h|e_1) < P'''(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $x_1, x_2 \in D_t$, if $x_1 > x_2$, then $t(x_1) < t(x_2)$. By a similar argument, A1 also implies that, for any $x_1, x_2 \in D_t$, if $x_1 = x_2$, then $t(x_1) = t(x_2)$. So, for A1 to hold, it must be that, for any $e, h \in L_c$ and any $P \in \mathbf{P}$, if $P(h|e) > P(h)$, then $C_P(e, h) = t[P(\neg h|e)/P(\neg h)]$ and t is a strictly decreasing function.

Summing up, if A0–A3, then for any $e, h \in L_c$ and any $P \in \mathbf{P}$, (i) if $P(h|e) \leqslant P(h)$, then $C_P(h, e) = m[P(h|e)/P(h)]$ and m is a strictly increasing function, thus $C_P(h, e)$ is a strictly increasing function of $z(h, e)$, and (ii) if $P(h|e) > P(h)$, then $C_P(h, e) = t[P(\neg h|e)/P(\neg h)]$ and t is a strictly decreasing function, thus $C_P(h, e)$ is a strictly increasing function of $z(h, e)$. As (i)–(ii) are exhaustive, for A0–A3 to hold, it must be that, for any $e, h \in L_c$ and any $P \in \mathbf{P}$, $C_P(h, e) = f[z(h, e)]$ and f is a strictly increasing function. □

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